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On a Special Elliptic Ruled Surface
of the Ninth Order

Dissertation.

Submitted to the Board of University Studies of the Johns Hopkins University, in conformity with the requirements for the degree of Doctor of Philosophy.

By
Harry Clinton Gossard

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To Professor Morley, without whose helpful suggestions and constant inspiration this paper would have been impossible, and to doctors Coble, Cohen, Hulburt and Bateman for their valuable instruction and inspiration in his university courses, the author desires to express his thanks.

On a Special Elliptic Ruled Surface of the Ninth Order

Introduction

The object of this paper is to discuss the following problem connected with a tetrahedron.

Are there lines connected with a tetrahedron, T , such that if the vertices are reflected in these lines, the reflections will fall on the opposite faces? If so what is the locus of these lines?

G.T. Bennett* has shown that for a chosen T , there are ∞^1 such lines and that when the opposite pairs of edges of T are equal, the locus consists of 3 cylindroids. First we shall establish the existence of these ∞^1 lines by direct geometric considerations, and then discuss their locus.

* Proc. Lon. Math. Soc. Series 2 - Vol. 10.
Parts 4 and 5. (1911).

§ 1. Collineation between vertices and opposite faces of a tetrahedron.

Let $(xx)(a\phi) = 0$, be a collineation between a point x , and a point x' . Then the coefficients of the ϕ 's can be taken as the coordinates of the point x' . This gives:

$$kx_0' = a_0(xx)$$

$$kx_1' = a_1(xx)$$

$$kx_2' = a_2(xx)$$

$$kx_3' = a_3(xx)$$

where k is the factor of proportionality. If we let $x_i' = x_i$, the eliminant of these equations will give the fixed points of the collineation. This eliminant is:

$$\begin{vmatrix} a_0x_0 - k & a_0x_1 & a_0x_2 & a_0x_3 \\ a_1x_0 & a_1x_1 - k & a_1x_2 & a_1x_3 \\ a_2x_0 & a_2x_1 & a_2x_2 - k & a_2x_3 \\ a_3x_0 & a_3x_1 & a_3x_2 & a_3x_3 - k \end{vmatrix} = 0,$$

or $k^4 - \tau_1 k^3 + \tau_2 k^2 - \tau_3 k + \tau_4 = 0$, where the τ 's have the following form.

$$J_1 = (ax) = u_0 x_0 + u_1 x_1 + u_2 x_2 + u_3 x_3$$

$$J_2 = (x_3 \cdot ab) = |x_3| |ab|,$$

$$J_3 = (x_3 v \cdot abc) = |x_3 v| |abc|$$

$$J_4 = (x_3 v \delta \cdot abcd) = |x_3 v \delta| |abcd|,$$

Suppose the above collineation to be a displacement. If this displacement is effected by a screw motion, we would have:

$$x' = e^{ix} x$$

$$\bar{x}' = e^{-ix} \bar{x}$$

$$z' = z + p x$$

where p is the pitch of the screw, x the unit turned turn, and the barred variable is the reflection of the unbarred variable in the axis z .

These equations written in vector-representations are:

$$x_0' = e^{ix} x_0$$

$$x_1' = e^{-ix} x_1$$

$$x_2' = x_2 + p_1 x_3$$

$$x_3' = x_3$$

$\mathcal{F}_1$ then is $e^{ix} + e^{-ix} + 1 + 1$

$$\text{or } \mathcal{F}_1 = 2 + 2 \cos x.$$

We shall now ask that this displacement send a vertex of the reference tetrahedron, which we shall designate as T , onto the face of opposite.

If the vertex, $(1, 0, 0)$, is to go onto the opposite face, $(0, 1, 1)$, then $x_1 = x_0$, and

$$u_1 = 0 \text{ and consequently } (x_1, u_1) = 0$$

becomes $x_0 x_0 = 0$. In the same way, we get for the collocation that send the three other vertices onto the

corresponding opposite faces; -

$$u_1 x_1 = 0, \quad u_2 x_2 = 0, \quad u_3 x_3 = 0$$

If all the vertices are required to go onto the opposite faces, simultaneously we shall have

$$* \mathcal{F} \equiv (u, x) = 0.$$

* The converse is easily seen to be true, i.e. if $\mathcal{F}_1 = 0$, the corresponding displacement sends the vertices on to the opposite faces.

For the screw displacement τ_1 where $\tau_1 = \angle \phi - \cos \phi$, we see that $\tau_1 = 0$ requires $\phi = 180^\circ$. Therefore from the above consideration we deduce this statement:

There are displacements of T which send each vertex onto the opposite face, and these displacements consist of rotations about an axis, thru an angle of 180° , plus a translation.

This statement is also substantiated by the constants involved. General collineation has 16 coefficients, and therefore involves 15 independent constants. In the displacement considered, we must have unaltered the plane at infinity, the absolute in this plane, and the size of T . This is $3 + 5 + 1$, or 9 conditions. $\therefore$ there are 6 conditions on each vertex. $\therefore$ 6

it lie on the face or outside it, or $\bar{7}$ in all. This leaves $15-9-4$ or 2 constants at our disposal, - a number just sufficient to determine the axes of the rotations.

Instead of thinking of a single tetrahedron, T , and its positions after the displacements, we shall call the displaced tetrahedra T' , and the original tetrahedron, T , where T is to be the reference tetrahedron through out this paper. Then T is inscribed to each T' , and T and T' are such tetrahedra that a rotation of T thru angle of 180° ^{plus a translation} would make T and T' coincident. The pitch of this screw motion will now be taken to be 0 , and then T' is a reflection of T in the axes of rotation

Ex 2. 170 - 171 -

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the roots of the polynomial

$$x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$$

the α_i are the roots of

$$R_0: \alpha_1 = -\alpha_2, \quad \alpha_1' = \alpha_2', \quad \alpha_1'' = -\alpha_2'', \quad \alpha_1''' = \alpha_2'''$$

$$R_1: \alpha_1 = \alpha_2, \quad \alpha_1' = -\alpha_2', \quad \alpha_1'' = \alpha_2'', \quad \alpha_1''' = -\alpha_2'''$$

$$R_2: \alpha_1 = \alpha_2, \quad \alpha_1' = \alpha_2', \quad \alpha_1'' = -\alpha_2'', \quad \alpha_1''' = -\alpha_2'''$$

$$R_3: \alpha_1 = \alpha_2, \quad \alpha_1' = -\alpha_2', \quad \alpha_1'' = \alpha_2'', \quad \alpha_1''' = \alpha_2'''$$

R_0 is the identity, R_1 is the transposition, R_2 is the

$$p_{11}, p_{12}, p_{21}, p_{22}, p_{31}, p_{32}, p_{33}, p_{34}$$

the α_i are the roots of the polynomial

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$$p_{11}, p_{12}, p_{21}, p_{22}, p_{31}, p_{32}, p_{33}, p_{34}$$

$$p_{11}, p_{12}, p_{21}, p_{22}, p_{31}, p_{32}, p_{33}, p_{34}$$

- γ will increase in α and β - γ is
 decreasing in α and β

ii) to meet the PR_0 and PR_1 i -

3) $p_{21} \bar{1} \bar{2} = p_{23} \bar{1} \bar{2} 3$, a linear temp. eq.

2. 1021 + 2 = 1023. 1023 is a prime number.

$$f(0, 1, 0) = 0, \text{ and } f(1, 0, 0) = 0.$$

12 11 10 9 8 7 6 5 4 3 2 1

$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$

1. 22th Feb. 1964. as seen by
observing that and it was the
roles

Let us use the same notation as in the previous problem. In this case the equations are

$$x_0(\pi_{01}\pi_{02}\pi_{03} - \pi_{10}\pi_{20}\pi_{30}) + \lambda_1^2\pi_{23} = 0$$

$$x_1(\pi_{12}\pi_{13}\pi_{21}\pi_{31} - \pi_{20}\pi_{30}\pi_{10}) + \lambda_2^2\pi_{31} = 0$$

$$x_2(\pi_{23}\pi_{21}\pi_{32}\pi_{12} - \pi_{30}\pi_{10}\pi_{20}) + \lambda_3^2\pi_{12} = 0$$

Let us take x_0, x_1, x_2, x_3 as unknowns.

$$x_0 = \frac{1}{\lambda_1^2} (\lambda_1^2 \pi_{01}\pi_{02}\pi_{03} + \lambda_1^2 \pi_{12}\pi_{13}\pi_{20} + \lambda_2^2 \pi_{20}\pi_{21}\pi_{23} + \lambda_3^2 \pi_{31}\pi_{32}\pi_{30})$$

$$\text{Similarly } x_1 = -\lambda_1^2$$

$$x_2 = -\lambda_2^2$$

$$x_3 = -\lambda_3^2$$

These equations are satisfied by $\lambda = i$ and also by i [Result 1]

$$x_0^2 \pi_{01}\pi_{02}\pi_{03} + \lambda_1^2 \pi_{10}\pi_{20}\pi_{30} + \lambda_2^2 \pi_{20}\pi_{21}\pi_{23} + \lambda_3^2 \pi_{30}\pi_{31}\pi_{32} = 0$$

As a consequence of the above, the

$$\frac{x_0^2 \pi_{01}\pi_{02}\pi_{03} + \lambda_1^2 \pi_{10}\pi_{20}\pi_{30} + \lambda_2^2 \pi_{20}\pi_{21}\pi_{23} + \lambda_3^2 \pi_{30}\pi_{31}\pi_{32}}{\lambda_1^2 \pi_{23} + \lambda_2^2 \pi_{31} + \lambda_3^2 \pi_{12}} = 0$$

$$\frac{x_0^2 \pi_{01}\pi_{02}\pi_{03} + \lambda_1^2 \pi_{10}\pi_{20}\pi_{30} + \lambda_2^2 \pi_{20}\pi_{21}\pi_{23} + \lambda_3^2 \pi_{30}\pi_{31}\pi_{32}}{\lambda_1^2 \pi_{23} + \lambda_2^2 \pi_{31} + \lambda_3^2 \pi_{12}} = 0$$

quadric, a point of contact with the plane at infinity, which we shall call y , and a line Π , which is the polar line of p , with regard to the absolute.

Now $(\frac{x^2}{y}) \equiv \frac{x_0^2}{\gamma_0} + \frac{x_1^2}{\gamma_1} + \frac{x_2^2}{\gamma_2} + \frac{x_3^2}{\gamma_3} = 0$, represents any quadric referred to a given self-polar tetrahedron as tetrahedron of reference. To make this quadric a paraboloid it must be required to touch the plane at infinity, which is $(x) = 0$. The condition for this is:

$$\frac{1}{\gamma_1 \gamma_2 \gamma_3} + \frac{1}{\gamma_0 \gamma_2 \gamma_3} + \frac{1}{\gamma_0 \gamma_1 \gamma_3} + \frac{1}{\gamma_0 \gamma_1 \gamma_2} = 0$$

or $(y) = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 = 0$.

Then $(\frac{x^2}{y}) = 0$ is to be orthic, i.e. must be polar to the absolute, as will be shown later, the equation of the absolute can be written as $(A\xi)^2 = 0^*$. For $(\frac{x^2}{y}) = 0$ to be polar to

$(A\xi)^2 = 0$, the necessary condition is:

$$(\frac{A^2}{y}) \equiv \frac{A_{00}}{\gamma_0} + \frac{A_{11}}{\gamma_1} + \frac{A_{22}}{\gamma_2} + \frac{A_{33}}{\gamma_3} = 0.$$

* where A_{ii} is the square of the area of the face of Π , which is opposite the vertex x_i .

Then all lines, p , lie on the quadrics represented by $(\frac{X^2}{y}) = 0$, where these quadrics are subjected to the ^{two} relations $(\gamma) = 0$ and $(\frac{A^2}{y}) = 0$. Evidently from the fore-going statements the axis of any one of these quadrics is on y , and as p is orthic with regard to Π and y is on Π , it can be seen that p is one of the principal generators of the quadric. Equally well can the other ^{principal} generator ^{seen to be} a line, p' , for which the corresponding Π' is on y and the former p . i.e.

all the lines sought, are the principal generators of the quadrics $(\frac{X^2}{y}) = 0$, where $(\gamma) = 0$ and $(\frac{A^2}{y}) = 0$.

a case in point is the rectangular ^{paraboloid} $xy = 4z$. The polar line of a point, x', y', z' is $xy' + x'y = z + z'$.

(12')

This equation is satisfied by

(1) $x', -y', -z'$, which is the reflection of x', y', z' in the line $x=0$.

(2) $-x', y', -z'$, which is the reflection of x', y', z' in the y -axis.

These two lines are the principal

generators of $xy = z$. If (x', y', z') be a vertex of π , and $xy' + x'y = z + z'$ be l.e. of opposite face of π then a reflection of this vertex in either of the above lines is a point of the face opposite this vertex. Thus if we choose three points $(a, b, c), (d, e, f)$ and (r, s, t) and get their reflection planes as to $xy = z$, they in turn will determine the fourth point which, with the given three points, determine a tetrahedron which is self polar with regard to $xy = z$. The plane on the above points is

$$(7) \begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \\ r & s & t \end{vmatrix} = 0 \quad \text{or} \quad Ax - By + Cz - D = 0$$

where A is the first minor of x , B of y , etc.

The three reflected points $(a, -b, -c), (d, -e, -f), (r, -s, -t)$ are on the line

$$(8) Ax + By - Cz - D = 0.$$

faces of π , in coplanar points.

and the three reflections in the axes are on the line

$$4: Ax + By + Cz + D = 0.$$

The polar point of the line 7 with regard to $xy = z^2$ is the point $(\frac{B}{C}, -\frac{A}{C}, \frac{D}{C})$ which is on the lines 8 and 9 as seen by substituting these values for x, y, z in their equations. Similarly the polar point of (8) is $(\frac{B}{C}, \frac{A}{C}, \frac{D}{C})$ which is on (7) and the polar point of (9) is $(-\frac{B}{C}, -\frac{A}{C}, \frac{D}{C})$ which is also on (7).

The polar planes of $(a, -c, -t)$ and $(x, -y, -z)$ with regard to $xy = z^2$ are respectively:

$$1' \quad bx - ay + z - c = 0$$

$$2' \quad ex - by + z - f = 0$$

$$3' \quad cx - ay + z - t = 0.$$

$(x, -y, -z)$ is on $1'$ and $(a, -c, -t)$ is on $2'$ if $bx + ay - z - c = 0$. $(x, -y, -z)$ is on $3'$ if $cx + ay - z - t = 0$.

$(a, -s, -t)$ is on $(2')$ and $(d, -e, -f)$ on $(3')$ if $a + ds - t - e = 0$. These three conditions make $(a, -s, -t)$, $(d, -e, -f)$, $(a, -s, -t) \sim d(\frac{B}{C}, \frac{A}{C}, \frac{D}{C})$ the four vertices of a self-polar tetrahedron, T . The same conditions make the reflections of these four vertices, in the x -axis, lie upon the faces of T and form a tetrahedron, T' to which T is circumscribed.

The same conditions make the reflections in the y -axis, form the tetrahedron, T'' , to which T is also circumscribed.

As there are only 3 conditions necessary to determine T' or T'' , it is at once seen that there are ∞^1 lines about which T can be rotated thru 180° and thus become coincident with the T' 's.

§5. Introduction of Elliptic Functions

$(\frac{x^2}{y}) = 0$, is a cubic surface containing the 6 edges of T . Consequently, when we make $(y) = 0$, and so get the intersection of this surface and the plane at infinity, we have a cubic curve which passes thru the 6 points where the edges of T meet this line. This enables us to introduce an elliptic parameter in terms of which we can express this cubic as well as other curves arising from later considerations.

Let the above 6 points be named x, y, z, u, v, w , as in the figure.

Then since these are elliptic parameters of the cubic and are by 3's on a line,

$$a+b+c=0, \quad b+x+y=0,$$

$$a+y+z=0, \quad c+x+z=0,$$

Adding the last three equations and substituting the first in the result, we have,

$x+y+z=0$, or ω_1 , where ω_1 is a half-period. Then

$$x = \omega_1 - y - z = \omega_1 + a$$

$$y = \omega_1 + b, \quad z = \omega_1 + c.$$

Consider $y_i = h_i \sigma(u-a) \sigma(u-b) \sigma(u-c)$

This represents a line picking out the three values a, b , and c , whose sum is 0.

The line on a, b, c , is $y_0 = h_0 \sigma(u-a) \sigma(u-b) \sigma(u-c)$

on a, y, z , $y_1 = h_1 \sigma(u-a) \sigma(u-b-\omega_1) \sigma(u-c-\omega_1)$

and similar equations for lines on b, y, z and c, x, z respectively.

The function $\sigma(u-b-\omega_1) = \sigma_1(u-b)$ where σ_1 is the allied σ function.
The above equations are the -

$$y_1 = \Lambda_1 \sigma_1(u-a) \sigma_1(u-b) \sigma_1(u-c)$$

$$1_1 = \Lambda_1 \sigma_1(u-a) \sigma_1(u-b) \sigma_1(u-c)$$

in which it is to be noted that $\sigma_1(u) = 1$ and $\sigma_1^2(u) = 1$.

$$\sigma_1(u) = 1$$

$$\Lambda_1 \sigma_1(u-a) \sigma_1(u-b) \sigma_1(u-c) + \Lambda_2 \sigma_1(u-a) \sigma_1(u-b) \sigma_1(u-c) + \Lambda_3 \sigma_1(u-a) \sigma_1(u-b) \sigma_1(u-c) = 1$$

Dividing by the coefficient of Λ_1 we get

$$(10) \Lambda_1 + \Lambda_2 \frac{\sigma_1(u-a) \sigma_1(u-b)}{\sigma_1(u-a) \sigma_1(u-b)} + \Lambda_3 \frac{\sigma_1(u-a) \sigma_1(u-b)}{\sigma_1(u-a) \sigma_1(u-b)} = 1$$

We shall now remove the variables u, a, b, c , by writing $u=a, u=b, u=c$ in turn in the above equation. (σ_1 in this case = 1, as $\sigma_1(0) = \sigma_1(-a) = 1$.)

This gives

$$\left[\Lambda_2 \frac{\sigma_1(u-a)}{\sigma_1(u-a)} + \Lambda_3 \frac{\sigma_1(u-b)}{\sigma_1(u-b)} \right] \frac{1}{u-a} = 1$$

$$\left[\Lambda_1 \frac{\sigma_1(u-a)}{\sigma_1(u-a)} + \Lambda_3 \frac{\sigma_1(u-b)}{\sigma_1(u-b)} \right] \frac{1}{u-b} = 1$$

$$\left[\Lambda_1 \frac{\sigma_1(u-a)}{\sigma_1(u-a)} + \Lambda_2 \frac{\sigma_1(u-b)}{\sigma_1(u-b)} \right] \frac{1}{u-c} = 1$$

and throwing out the factors $\frac{1}{u-a}$ etc. we remove the values for u which

Equation 2 would need to be solved for λ_1 and λ_2

$$\lambda_1 \frac{\sigma_1(u-e)}{\sigma_1(u-e)} = 0 \quad \text{or} \quad \lambda_1 = \frac{\sigma_1(u-e)}{\sigma_1(u-e)}$$

$$\lambda_2 \frac{\sigma_2(u-e)}{\sigma_2(u-e)} = 0 \quad \lambda_2 = \frac{\sigma_2(u-e)}{\sigma_2(u-e)}$$

$$\lambda_3 \frac{\sigma_3(u-e)}{\sigma_3(u-e)} = 0 \quad \lambda_3 = \frac{\sigma_3(u-e)}{\sigma_3(u-e)}$$

Substituting these values into (1) we get

then

$$(1) \quad \lambda_0 + \sigma_1(u-e) \sigma_1(u-e) + \dots + \dots = 0$$

To determine λ_0 , let $u = u_0$

so $u-u_0 = 0$ and $\sigma_1(u-e) = \sigma_1(u_0-e) = 0$

But since $\sigma_2(u-e) \neq 0$ at $u = u_0$

we get from (1) $\lambda_0 = -\sigma_2(u_0-e)$

$$(12) \quad \lambda_0 + \sigma_1(u-e) \sigma_1(u-e) + \dots + \dots = 0$$

Now

$$\frac{\sigma_1(u-b)}{\sigma(u-b)} = \sqrt{p(u-b)-e_1}$$

$$\frac{\sigma_1(u-b+\omega_1)}{\sigma(u-b+\omega_1)} = \sqrt{p(u-b+\omega_1)-e_1}$$

$$\begin{aligned} \frac{\sigma_1(u-b)}{\sigma(u-b)} \frac{\sigma_1(u-b+\omega_1)}{\sigma(u-b+\omega_1)} &= \sqrt{p(u-b)-e_1} \sqrt{p(u-b+\omega_1)-e_1} \\ &= \sqrt{e_3 - e_1 e_2 - e_1} \end{aligned}$$

$$\frac{\sigma_1(u-v)}{\sigma_1(u-v)} = \frac{\sigma_1(u-v)}{\sigma_1(u-v)} \cdot \frac{\sigma_1(v-w)}{\sigma_1(v-w)} = \frac{\sigma_1(u-w)}{\sigma_1(u-w)}$$

and as $\frac{\sigma_1(u-b+w_1)}{\sigma_1(u-b+w_1)} = \sqrt{(l_1-l_2)(l_1-l_3)} \frac{\sigma_1(u-b)}{\sigma_1(u-b)}$

$$\frac{\sigma_1(u-b)}{\sigma_1(u-b)} = \frac{\sigma_1(a-b)}{\sigma_1(a-b)} \sqrt{(l_1-l_2)(l_1-l_3)}$$

Similarly

$$\frac{\sigma_1(u-c)}{\sigma_1(u-c)} = \frac{\sigma_1(a-c)}{\sigma_1(a-c)} \sqrt{(l_1-l_2)(l_1-l_3)}$$

Substituting these values in (12), we get:

$$\Delta_0 = (l_1-l_2)(l_1-l_3) \frac{\sigma_1(b-c)\sigma_1(c-a)}{\sigma_1(b-c)\sigma_1(c-a)} \frac{\sigma_1(u-b)}{\sigma_1(u-b)}$$

Therefore (11) becomes:

$$(13) \quad (l_1-l_2)(l_1-l_3) \frac{\sigma_1(b-c)\sigma_1(c-a)}{\sigma_1(b-c)\sigma_1(c-a)} \frac{\sigma_1(u-b)}{\sigma_1(u-b)} + \sum \frac{\sigma_1(b-c)\sigma_1(u-b)}{\sigma_1(b-c)\sigma_1(u-b)} \frac{\sigma_1(u-c)}{\sigma_1(u-c)} = 0$$

This equation is the one corresponding to $y = 0$, and so

$$(14) \quad \begin{cases} y_0 = \frac{\sigma(b-c)\sigma(c-a)\sigma(a-b)}{\sigma_1(b-c)\sigma_1(c-a)\sigma_1(a-b)} (l_1-l_2)(l_1-l_3) \\ y_1 = \frac{\sigma(b-c)\sigma_1(a-b)\sigma(a-c)}{\sigma_1(b-c)\sigma_1(a-b)\sigma_1(a-c)} \end{cases}$$

and similar expressions for y_2 and y_3 .

In the last three terms of this equation, let us substitute $a + u$ for a . Since

$$\frac{\sigma_1(a-b)}{\sigma(a-b)} = \frac{\sigma(a-b)}{\sigma_1(a-b)} (l_1-l_2)(l_1-l_3)$$

this gives

$$(15) \quad (l_1-l_2)(l_1-l_3) \frac{\sigma(b-c)\sigma(c-a)\sigma(a-b)}{\sigma_1(b-c)\sigma_1(c-a)\sigma_1(a-b)} +$$

$$+ \sum_{i=2}^3 \frac{\sigma(b-c)}{\sigma_1(b-c)} (l_1-l_i)(l_i-l_j) \frac{\sigma(a-b)}{\sigma_1(a-b)}$$

$$+ (l_1-l_2)(l_1-l_3) \frac{\sigma(a-c)}{\sigma_1(a-c)} = 0$$

and dividing by $(l_1-l_2)(l_1-l_3)$

$$(15) \quad \frac{\sigma(b-c)\sigma(c-a)\sigma(a-b)}{\sigma_1(b-c)\sigma_1(c-a)\sigma_1(a-b)} + \sum_{i=2}^3 \frac{\sigma(b-c)\sigma(a-b)\sigma(a-c)}{\sigma_1(b-c)\sigma_1(a-b)\sigma_1(a-c)} = 0$$

2)

By a proper substitution of the values of the y 's of (1) in this equation, the result is:

$$\frac{\sigma^2(b-c)\sigma^2(c-a)\sigma^2(a-b)}{\sigma_1^2(b-c)\sigma_1^2(c-a)\sigma_1^2(a-b)} (l_1-l_2)(l_1-l_3) \frac{1}{y_0} + \sum_{i=1}^3 \frac{\sigma^2(b-c)}{\sigma_1^2(b-c)} \frac{1}{y_i} = 0$$

This is manifestly of the form

$$\left(\frac{A^2}{y}\right) = \frac{A_{00}}{y_0} + \frac{A_{11}}{y_1} + \frac{A_{22}}{y_2} + \frac{A_{33}}{y_3} = 0,$$

Where $A_{00} = (l_1-l_2)(l_1-l_3) \frac{\sigma^2(b-c)\sigma^2(c-a)\sigma^2(a-b)}{\sigma_1^2(b-c)\sigma_1^2(c-a)\sigma_1^2(a-b)}$

$$A_{11} = \frac{\sigma^2(b-c)}{\sigma_1^2(b-c)}$$

$$A_{22} = \frac{\sigma^2(c-a)}{\sigma_1^2(c-a)}$$

$$A_{33} = \frac{\sigma^2(a-b)}{\sigma_1^2(a-b)}$$

$$A_{00} = A_{11} A_{22} A_{33} (l_1-l_2)(l_1-l_3)$$

$$\text{and } y_0^2 = A_{00} (l_1-l_2)(l_1-l_3)$$

§4.9. The Plane at Infinity

The reflection of a vertex, V_i of Π in any line at infinity will lie on the face, F_i , of Π . For, to get a reflection of a point in two lines we take the fourth harmonic of this point and the two intersections of the two given lines with the line that can be drawn thru the given point and the two lines.

Now if we take it as any line in W ,* and p' is its polar point - as to be absolute, then any line on p' will be

for the record of the lines V_i & reflection. In particular we can take the line p' that passes thru the intersection of Π and F_i . Then the reflections of V_i are
 * Hereafter we shall call the plane at infinity, W .

be at the intersection of the lines Π and Σ . The line Σ is then to be included in the line locus Σ . So Σ of the lines in Π , though, we in a special way, lines of Σ , as we shall now see.

First let us find the locus in Π of the lines Π , as defined in Σ . For each Π , there is a point y on it, and for all the quadrics which are orthic paraboloids and self polar with regard to T , the corresponding y 's lie on the cubic curve, $(\frac{x^2}{y}) = 1$. We shall now get the locus of all the lines in Π , corresponding to the y 's.

The two principal generators of any of these quadrics are perpendicular to each other and to

the set of P and Q is, P is
 axis, and Q is P .

So in P we

have this geomet-

rical relation:

P , one of the two P .

generators 2 and 1 , the

vertex of the quadric,

directly it is point 1 and 2 on

the Q is point 1 and 2 on

the Q . Because Q is a line and

is finite, and Q is finite, the point 1 ,

is the other point 2 on Q ,

and Q is a line and Q is finite,

and Q is a line and Q is finite,

and Q is a line and Q is finite,

and Q is a line and Q is finite,

and Q is a line and Q is finite,

and Q is a line and Q is finite.

Regarding the point-pairs, that arise from the principal generators of the quadrics here considered, it has already been stated that they are on two lines, π and π' , that meet on the cubic curve, $(\frac{A^2}{y})=0$. These line pairs are conjugates as to the absolute and form the degenerate conics of the net of apolar conics of ~~the same~~ ^{cubic} ~~absolute~~. The cubic, $(\frac{A^2}{y})$, is then the Jacobian of this net and the Hessian of the cubic(s) of which this net is the polar net.* The

Cayleyan of the latter cubic would be enveloped by these line pairs, and therefore the reciprocal of this Cayleyan, as to the absolute, will pass through the point-pairs, as v_1 and v_2 . as lines of Ω pass through these points, this cubic will be at least a part

* Clifford's Papers, "Polar Theory of the Cubic Curve."

of the intersection of Σ and W .

The complete intersection is a 9-ic, ^{as will be shown later} and there is yet a sextic to account for. This sextic consists of the 6 tangents to the above cubic at its intersections with the absolute.

These 6 lines, p , will be paired off with the 6 lines, π , that are tangents to the absolute at these intersections.

The p 's that intersect W at these intersections will be perpendicular to the above 6 lines, p , and thus we see that these 6 points are the intersections of W and the sextic double curve of Σ .

The plane 9-ic cut out of Σ by any plane, will be an elliptic 9-ic and consequently must have 27 double points. In the plane W , we have these double points in

evidence. They are the 15 sheets of the 6 lines; the 3 points where the cubic meets the absolute; and the 3 points where the tangents of the cubic (at its intersection with the absolute) again meet the cubic.

Since the cubic ($\frac{A^2}{y}$) is the Hessian of, and the cubic, above, is the reciprocal of the Cayleyan of the same cubic, ^{the coefficients of} this reciprocal cubic should be expressible in terms of the coefficients of $\frac{A^2}{y} = 0$, i.e. in terms of the areas of the faces and the lengths of the edges of T .

§ . Special Lines of Ω .

Bennett* has mentioned the following lines.

(1) The two perpendicular normals to an edge at its middle point, each equally inclined to the faces through the edge; i.e. lying in the planes that bisect the dihedral angles on an edge.

(2) Any line which can be drawn meeting two opposite edges and bisecting, internally or externally, at each of its extremities, the angle subtended by the opposite edge.

It is seen from ^{simple} geometric observations that these lines are proper lines of Ω . The lines (1) give a double-point of the surface at each mid-edge of T , and as the generators that meet in these points

* Proc. Lon. Math. Soc. series 2, 24, 10 p. 1

perpendicular pairs, these points are on the double-curve (C).

For each pair of opposite edges $op T$, there are seven such lines as (2). This is seen from the

correspondence set up. For if 12 and 34 are opposite edges of T , then for two points 1 and 2 on one line, we have one y on the other line and

this point y , determines a definite point x on the line on 1 and 2 . Thus x , in turn, determines 2 points 3 , and 4 on the 2nd line. So our correspondence is of such a nature that 2 x 's pick out 2 y 's and y 's pick out 2 x 's, i.e., the correspondence is 4-to-4, and therefore has 1+4 or 5 coincidences. — there are 3 lines of the type 2, 1. since the lines

at infinity is to be ~~included~~ out
 in the consideration of Ω , the
 line at infinity on the two sheets
 must not be counted. ~~the lines~~
 1 such lines of Ω on such a Σ
 of the tetrahedron of intersection.
 These with the pair of lines
 on each sheet make a total of 4
 lines on each side of it. There are
 two vertices, then 1. of these lines.

This shows that the intersection
 Ω is 1, since a line meets it in
 1 point. Hence Ω , we can write
 the equation of Ω in Σ this new
 points i.e. we can express the coordi-
 nates of any line of Ω in terms of
 an elliptic parameter.

To do this, we will suppose
 that the intersection cubic of Ω and
 W is expressed in terms of an elliptic

parameter, u . Then since all lines of Ω meet the cube, any one of these lines can be named in terms of the same parameter u , there being factors like $\sigma(u-a)$ for each line that meets an edge of the tetrahedron of reference, and so we have

$$p_{ij} = \sigma(u-a_{ij}) \sigma(u-b_{ij}) \cdots \sigma(u-k_{ij})$$

For the pair of perpendicular lines at the mid-points of an edge, the parameters would differ only by a period - i.e. if one is $u-\alpha$, the other is $u-\alpha-\omega$. Then we would have

$$p_{ij} = \sigma(u-a_{ij}) \sigma(u-b_{ij}) \sigma(u-c_{ij}) \\ \sigma(u-d_{ij}) \sigma(u-e_{ij}) \sigma(u-f_{ij}) \\ \sigma(u-g_{ij}) \sigma(u-h_{ij}) \sigma(u-i_{ij}-\omega)$$

Moreover, since the st 7 of these factors represent the 7 lines that meet the cube at its corners, i.e. are

factors of i_j that meet it, let
the edge (i, j) will equally meet it
the same way meet the edge it.

So if $p_{01} = \sigma(u-a_{01}) \cdots \sigma(u-g_{01}) \sigma(u-h_{01}) \sigma(u-h_{01}-w)$

then $p_{23} = \sigma(u-a_{01}) \cdots \sigma(u-g_{01}) \sigma(u-h_{23}) \sigma(u-h_{23}-w)$

and similarly for $p_{0-1}, p_{12}, p_{23}, p_{34}, \dots, p_{r+2}$

§. 1 The Double Curve

It has already been stated that the lines of the surface Ω , pair off in such a way that, ^{the lines of} each pair are the principal generators of an orthic paraboloid. Therefore the surface Ω has a double point curve on it; — the intersection of the lines in each pair, — which is ^{the vertices of} the locus of all the orthic paraboloids of the system, i.e. of all such quadrics that have Π as their tetrahedron of reference, where Π is self-polar. In each line of Ω is a tangent plane of Ω , and so the tangent planes of each pair of lines coincide in the plane tangent to the one of the quadrics (at its vertex,) of which ^{the lines of the} pair are the principal generators. Then consequently,

there will be a double-plane curve which these planes will envelop. We shall now proceed to find the equations of these two double curves, getting the double curve in planes first.

The points y and the lines, ~~of~~ ^{of the} Σ , are pole and polar as to the absolute and it has already been stated that the two principal generators of the paraboloid that touches the plane at infinity in this point y , meet Π in points that are conjugates as to the absolute. So to get the plane that is tangent to the paraboloid at the vertex, we shall take the polar plane of y as to the circumsphere of Π , and then take the plane parallel to this that touches the quadric.

The circum-sphere is given in quadri-planar coordinates as:

$$\frac{l_{21}^2 \bar{x}_1 \bar{x}_2}{x_1' x_2'} + \frac{l_{20}^2 \bar{x}_2 \bar{x}_0}{x_2' x_0'} + \frac{l_{01}^2 \bar{x}_0 \bar{x}_1}{x_0' x_1'} + \frac{l_{03}^2 \bar{x}_0 \bar{x}_3}{x_0' x_3'} + \frac{l_{13}^2 \bar{x}_1 \bar{x}_3}{x_1' x_3'} + \frac{l_{23}^2 \bar{x}_2 \bar{x}_3}{x_2' x_3'} = 0 \quad *$$

where l_{ij} is an edge of Π , and $x_0' x_1' x_2' x_3'$ are the altitudes of Π , to

* Rogers' revision of Salmon's Geom. of 3 dimensions, p 235

the corresponding faces $x_0=0, x_1=0, \dots$
^{Barycentric}
 x_n projective coordinates, this is
 easily seen to be:

$$l_{21}^2 x_2 x_1 + l_{20}^2 x_2 x_0 + l_{01}^2 x_0 x_1 + l_{03}^2 x_0 x_3 \\ + l_{13}^2 x_1 x_3 + l_{23}^2 x_2 x_3 = 0$$

$$\text{or } \sum_{i,j} l_{ij}^2 x_i x_j = 0, (i=0,1,2,3, j=0,1,2,3)$$

The polar plane of y , as to this sphere
 is: $\sum_{i,j} l_{ij}^2 (x_i y_j + x_j y_i) = 0$

which can be written

$$\sum_{i,j} (l_{ij}^2 y_j + l_{ik}^2 y_k + l_{im}^2 y_m) x_i = 0 \text{ or } (C x) = 0$$

(where $j \neq k \neq m \neq i$)

Any plane parallel to $(C x) = 0$ would be
 (17) represented by: $(C_0 + \lambda) x_0 + (C_1 + \lambda) x_1 \\ + (C_2 + \lambda) x_2 + (C_3 + \lambda) x_3 = 0.$

The condition for this plane to touch
 the quadric $(\frac{x^2}{y}) = 0$, is:

$$\frac{(C_0 + \lambda)^2}{y_1 y_2 y_3} + \frac{(C_1 + \lambda)^2}{y_0 y_2 y_3} + \frac{(C_2 + \lambda)^2}{y_0 y_1 y_3} + \frac{(C_3 + \lambda)^2}{y_0 y_1 y_2} = 0$$

$$\text{or } (C_0 + \lambda)^2 y_0 + (C_1 + \lambda)^2 y_1 + (C_2 + \lambda)^2 y_2 + (C_3 + \lambda)^2 y_3 = 0$$

Which in terms of λ is

$$\lambda^2(\gamma) + 2\lambda(c\gamma) + (c^2\gamma) = 0.$$

The γ 's, we recall, are to satisfy the two relations $(\gamma) = 0$, and $(\frac{A^2}{\gamma}) = 0$.

$(c^2\gamma)$ consists (when the values of c are substituted) of 4 terms like:

$$\gamma_1\gamma_2\gamma_3 [2(l_{12}^2 l_{23}^2 + l_{23}^2 l_{31}^2 + l_{31}^2 l_{12}^2) - l_{12}^4 - l_{13}^4 - l_{23}^4]$$

and it is seen at once that this coefficient of $\gamma_1\gamma_2\gamma_3$ is $16A_{00}$.

Then $(c^2\gamma) = 16\gamma_0\gamma_1\gamma_2\gamma_3(\frac{A^2}{\gamma})$ and consequently $= 0$.

The above equation in λ then reduces to: $2\lambda(c\gamma) = 0$ which has the roots $\lambda = 0, \infty$. $\lambda = \infty$ substituted in equation (17') gives $(X) = 0$, the plane at infinity. $\lambda = 0$, gives the plane $(cX) = 0$ which then must be the tangent plane at the vertex, and as γ traces out the cubic curve, $(\frac{A^2}{\gamma}) = 0$, in the plane

at infinity, $(CX) = 0$ envelopes the
vertices of the paraboloids correspond-
ing to the y 's.

The equation of this double curve
 (in planes) is given at once by sub-
 stituting for the y 's the values given
 by (14) in § 3. Let us condense these values
 by the following substitutions:

$$A = \sigma(b-c) \quad U_a = \sigma(u-a)$$

$$B = \sigma(c-a) \quad U_b = \sigma(u-b)$$

$$C = \sigma(a-b) \quad U_c = \sigma(u-c)$$

$$A_1 = \sigma_1(b-c) \quad U_{1a} = \sigma_1(u-a)$$

etc.

etc.

Then

$$\begin{aligned} (CX) = & (\ell_{20}^2 A_1 B C_1 U_{1a} U_{1b} U_{1c} + \ell_{20}^2 A B_1 C_1 U_a U_b U_c \\ & + \ell_{30}^2 A_1 B_1 C U_{1a} U_{1b} U_c) X_0 \\ & + (\ell_{01}^2 A B C \ell_1 \ell_2 \ell_3 U_a U_b U_c + \ell_{21}^2 A_1 B C_1 U_{1a} U_b U_c \\ & + \ell_{31}^2 A_1 B_1 C U_{1a} U_{1b} U_c) X_1 + \dots = 0. \end{aligned}$$

This is a cubic in σ and so the
tangent planes at the vertices en-
velop as a double-curve, this cubic.

If we call this envelope $\xi x = 0$, then the role of ξ as to the quadric $(x^2/y) = 0$, which in planes is $(\xi^2 y) = 0$, is found at once by taking as the coordinates of this polar point, x , (of ξ) the coefficients of ξ' in $(\xi \xi' y) = 0$ or $x_i = \xi_i y_i$. If x is the vertex of the quadric, say v , then

$V_i = \xi_i y_i$ represents the double point curve of Ω that is the intersection of the perpendicular pairs of generators. This is seen at once to be a sextic curve, and its equation is

$$\begin{aligned} V_0 = C_0 y_0 &= (l_{10}^2 y_1 + l_{20}^2 y_2 + l_{30}^2 y_3) y_0 \\ &= \left[l_{10}^2 \frac{A^2 B \cdot C}{A^2 B_1 C_1} \frac{U_1 b U_c}{U_b U_c} + l_{20}^2 \frac{A B C^2}{A_1 B_1 C_1^2} \frac{U_1 a U_c}{U_a U_c} \right. \\ &\quad \left. + l_{30}^2 \frac{A \cdot B C^2}{A_1 B_1 C_1^2} \frac{U_1 a U_b}{U_a U_b} \right] (l_1 - l_2)(l_1 - l_3) \end{aligned}$$

$$\begin{aligned} V_1 = C_1 y_1 &= l_{10}^2 y_0 y_1 + l_{12}^2 y_1 y_2 + l_{13}^2 y_1 y_3 \\ &= (l_1 - l_2)(l_1 - l_3) l_{10}^2 \frac{A^2 B \cdot C}{A^2 B_1 C_1} \frac{1/b \cdot 1/c}{1/b \cdot 1/c} \end{aligned}$$

$$+ l_{12}^2 \frac{AB}{A_1 B_1} \frac{U_a U_b U_c^2}{U_a U_b U_c^2} + l_{13}^2 \frac{AC}{A_1 C_1} \frac{U_a^2 U_b U_c}{U_c^2 U_b U_c}$$

and similar expressions for l_2 and l_3 .

Multiplying by $A_1 B_1 C_1 U_a^2 U_b^2 U_c^2$ we have:

$$20) V_0 = (l_1 - l_2)(l_1 - l_3) \left[A_{11} l_{10}^2 A_1 B_1 C_1 U_a^2 U_b U_c U_b U_c + A_{22} l_{20}^2 A_2 B_1 C_1 U_a U_b^2 U_c U_a U_c + A_{33} l_{30}^2 A_3 B_1 C_1 U_a U_b U_c^2 U_a U_c \right]$$

$$V_1 = (l_1 - l_2)(l_1 - l_3) \left[A_{11} l_{10}^2 A_1 B_1 C_1 U_a^2 U_b U_c U_b U_c + l_{12}^2 A_1 B_1 C_1 U_a U_b U_a U_b U_c^2 + l_{13}^2 A_1 B_1 C_1 U_b U_c U_a^2 U_b U_c \right]$$

and similar expressions for V_2 and V_3 .

This sextic is only a part of the double-point curve, for the surface being represented by an equation of the ninth degree in an elliptic parameter, any plane section would be an elliptic 9-ic. and as such would necessarily have 27 double points. Therefore, the complete double-point curve would be of degree 27 and the double-plane curve being of degree

three less, would be of degree 24.
 Then the complete double-point curve consists of a sextic, representing the intersections of those generators that meet in perpendicular pairs, and the rest of the ^{of the} generators intersections forming a curve of degree 21.
 There are 7 double points on every generator of Ω .

An interesting property of the planes on the perpendicular pairs of generators of Ω , i.e. those planes that envelop the cubic $(C) = 0$, is established as a result of the fact that these planes are the reciprocals of the points, y , on the cubic $(A_y^2) = 0$, at infinity, with regard to the circumsphere of T , - a relation before mentioned. The points of any plane cubic reciprocate, with regard to a sphere, into planes

on a cubic cone, that is of class 3.
 as the plane is the plane at infinity, and the reciprocal of this plane is the centre of the sphere, we have this fact; - all the planes touching the double-plane curve, (), pass thru the centre of the sphere, that is the circumsphere of I.

§.7. Relation between the Faces & Edges of a Tetrahedron

It is desirable to find the relations between the edges and the faces of T , in order to express the coefficients, in the foregoing expressions, entirely in terms of the areas of the faces. To find all such relations, we shall express the absolute in terms of the edges of T , and also in terms of the faces of T , & then equate coefficients.

First, let us find the equation of the absolute in the plane, in terms of the lengths of the edges of the fundamental triangle of reference.

Let e_0, e_1, e_2 be this triangle, and let the edges be E_0, E_1, E_2 , where E_i is the side opposite the vertex e_i . Also let the angle at e_i be α_i and δ_i and

δ'_i be the perpendicular distances from ε_i to the lines δ and δ' ; where δ and δ' are the Cartesian axes of reference.

Then the area of the triangle of reference

$$\text{is : } \Delta = \frac{1}{2} \begin{vmatrix} \delta_0 & \delta_1 & \delta_2 \\ \delta'_0 & \delta'_1 & \delta'_2 \end{vmatrix}$$

$$\text{or, } 2\Delta = -\delta_0(\delta'_2 - \delta'_1) + \delta_1(\delta'_2 - \delta'_0) - \delta_2(\delta'_1 - \delta'_0)$$

Now let φ_i be the ^{angle} included between δ and ε_i where δ is to be taken as the initial line and the angle taken in a positive direction. This gives:

$$\text{or } 2\Delta = \delta_0 \varepsilon_0 \cos \varphi_0 + \delta_1 \varepsilon_1 \cos \varphi_1 - \delta_2 \varepsilon_2 \cos \varphi_2$$

$$\text{also } \begin{vmatrix} 1 & 1 & 1 \\ \delta_0 & \delta_1 & \delta_2 \\ \delta_0 & \delta_1 & \delta_2 \end{vmatrix} = 0$$

$$\text{or, } \delta_0(\delta_2 - \delta_1) + \delta_1(\delta_0 - \delta_2) - \delta_2(\delta_0 - \delta_1) = 0$$

$$(2) \text{ or, } \delta_0 \varepsilon_0 \sin \varphi_0 + \delta_1 \varepsilon_1 \sin \varphi_1 - \delta_2 \varepsilon_2 \sin \varphi_2 = 0$$

Squaring (21) and (22) and adding we have:

$$4\Delta^2 = \delta_0^2 \varepsilon_0^2 + \delta_1^2 \varepsilon_1^2 + \delta_2^2 \varepsilon_2^2 - 2\delta_1 \delta_2 \varepsilon_1 \varepsilon_2$$

$$\cos(\varphi_1 - \varphi_2) - 2\delta_2 \delta_0 \varepsilon_2 \varepsilon_0 \cos(\varphi_2 - \varphi_0) + 2\delta_0 \delta_1 \varepsilon_0 \varepsilon_1 \cos(\varphi_0 - \varphi_1)$$

$$\varphi_2 - \varphi_1 = \alpha_0, \quad \varphi_0 - \varphi_1 = 180^\circ - \alpha_2, \quad \varphi_0 - \varphi_2 = \alpha_1$$

$$\text{Therefore } 4\Delta^2 = \delta_0^2 \varepsilon_0^2 + \delta_1^2 \varepsilon_1^2 + \delta_2^2 \varepsilon_2^2 - 2\delta_1 \delta_2 \varepsilon_1 \varepsilon_2$$

$$\cos \alpha_0 - 2\delta_2 \delta_0 \varepsilon_2 \varepsilon_0 \cos \alpha_1 - 2\delta_0 \delta_1 \varepsilon_0 \varepsilon_1 \cos \alpha_2$$

If δ and δ' are not perpendicular but form an angle, ω , with each other, the above equation becomes:

$$4\Delta^2 \cos \omega = \delta_0 \delta_0' \varepsilon_0^2 + \delta_1 \delta_1' \varepsilon_1^2 + \delta_2 \delta_2' \varepsilon_2^2 - \varepsilon_1 \varepsilon_2 (\delta_1 \delta_2' + \delta_1' \delta_2) \cos \alpha_0 - \varepsilon_2 \varepsilon_0 (\delta_2 \delta_0' + \delta_2' \delta_0) \cos \alpha_1 - \varepsilon_0 \varepsilon_1 (\delta_0 \delta_1' + \delta_0' \delta_1) \cos \alpha_2$$

The δ 's are proportional to the coordinates, ξ and ξ' of the lines δ and δ' . Substituting ξ_i and ξ_i' for δ_i and δ_i' and factoring we have:

$$4\Delta^2 \cos \omega = (\varepsilon_0 \xi_0 l^{\alpha_0} + \varepsilon_1 \xi_1 l^{\alpha_1} + \varepsilon_2 \xi_2 l^{\alpha_2}) (\xi_0'$$

$$z_0 e^{-i\alpha_0} + \xi_1' z_1 e^{-i\alpha_1} + \xi_2' z_2 e^{-i\alpha_2} \quad \text{or}$$

$$4A^2 = \frac{1}{\cos \omega} (\xi z e^{i\alpha} (\xi' z e^{-i\alpha}))$$

$$\text{and } 4A^2 = \infty \text{ for } \omega = 90^\circ$$

(23) Therefore $(\xi z e^{i\alpha})(\xi' z e^{-i\alpha}) = 0$ is the
equation of the absolute.

This is also verified to be the
absolute, thus:

$$E_0^2 = E_1^2 + E_2^2 - 2E_1 E_2 \cos \alpha_0$$

$$A = \frac{E_1 E_2 \sin \alpha_0}{2}$$

$$\text{and } \cotangent \alpha_0 = \frac{E_1^2 + E_2^2 - E_0^2}{4A} = \text{say } C_0$$

with similar expressions for C_1 and C_2 ,
also $C_1 + C_2 = \frac{E_0^2}{2A}$, $C_2 + C_0 = \frac{E_1^2}{2A}$, $C_0 + C_1 = \frac{E_2^2}{2A}$
Substituting these values for the C_i 's
in (23) we get:

$$(C_1 + C_2)\xi_0^2 + (C_2 + C_0)\xi_1^2 + (C_0 + C_1)\xi_2^2 \\ - 2C_0\xi_1\xi_2 - 2C_1\xi_2\xi_0 - 2C_2\xi_0\xi_1 = 0.$$

This is easily seen to represent
a pair of points on the line at infinity,
i.e. is the absolute in the plane.

Equation (23) can be written

the following form:

(27) $\sum \varepsilon_i \varepsilon_j \{i\} \{j\} \cos(ij) = 0$ where i and j can equal 0, 1, 2, ..., and $\cos(ij)$ stands for cosine of the angle between ε_i and ε_j .

There are two natural extensions of (27) which equally well would represent the absolute in space. If

we represent the four vertices of the tetrahedron of reference by l_0, l_1, l_2 , and l_3 , and the face opposite l_i by f_i , (i.e. the area of this face is f_i) the space analogue of (27) would be:

(28) $\sum_{i,j} f_i f_j \{i\} \{j\} \cos(ij) = 0$. Here $\{i\}$ and $\{j\}$ are line coordinates and the angle (ij) is the dihedral angle between the faces f_i and f_j . (i and $j = 0, 1, 2, 3$). Let A_{ij} represent $f_i f_j \cos(ij)$, and equation

(25) takes the more simple form:

$$(26) (A\{\})^2 \equiv A_{00}\{0\}^2 + A_{11}\{1\}^2 + A_{22}\{2\}^2 + A_{33}\{3\}^2 \\ + A_{01}\{0\}\{1\} + A_{02}\{0\}\{2\} + \dots + \dots = 0.$$

As a second extension of (24), let us write a similar equation of such kind that it will represent the absolute in each face that is a side of T . A_{ij} as defined above is, for every face, the square of the area of that face f_{ij} or, $A_{ii} = f_{ii}^2 \cos^2(i|i) = f_{ii}^2$, and so we will speak of A_i as the area of the face f_i . Also let l_{ij} be the edge of T that passes thru the vertices i and j . Then in each of the faces, the absolute is:

$$\begin{aligned} A_0: \sum_{i,j=0}^3 \varepsilon_i \varepsilon_j \{i\}\{j\} \cos(ij) &= 0, (i \text{ and } j = 1, 2, 3) \\ A_1: \sum_{i,j=0}^3 &= 0 \quad (\quad = 0, 2, 3) \\ A_2: \sum_{i,j=0}^3 &= 0 \quad (\quad = 1, 3) \\ A_3: \sum_{i,j=0}^3 &= 0 \quad (\quad = 0, 1, 2) \end{aligned}$$

The equation that it includes

these four is:

$$(27) \sum_{i,j=0}^3 \epsilon_i \epsilon_j \cos(\alpha_{ij}) = 0, \quad (i \text{ and } j = 0, 1, 2, 3)$$

where ϵ_i is the edge of T opposite the vertex, e_i , of each face that is on e_i .

(The slight difficulty in defining the edges ϵ_i will be removed in the next form in which the above equation is written, by the substitution of ϵ_{ij} for ϵ_k .)

We shall next express equations (26) and (27) in line coordinates, Π_{ij} , where by Π_{ij} we mean the minors of the matrix

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

First let us determine these coordinates for a line in each of the reference planes. The line Π is taken as an axis and ζ and η are planes on this axis, and if Π is in a reference plane, then the planes ζ and η will

being ~~in~~ on this ~~line~~, ~~in~~ the coordinates ξ_i and η_i will be line coordinates in this plane, where η is one of the ^{reference} planes.

In the plane

$$A_0 (\eta_0 = 0) \quad A_1 (\eta_1 = 0)$$

$$\pi = \begin{vmatrix} \xi_0 & \xi_1 & \xi_2 & \xi_3 \\ 1 & 0 & 0 & 0 \end{vmatrix} \quad \pi = \begin{vmatrix} \xi_0 & \xi_1 & \xi_2 & \xi_3 \\ 0 & 1 & 0 & 0 \end{vmatrix} \quad \text{etc.}$$

or from these matrices, and the quadratic identity between the plane and point coordinates of a line, we have:

in plane A_0 ; in A_1 ;

$$\xi_1 = \pi_{10} = \rho p_{32} \quad \xi_0 = \pi_{01} = \rho p_{23}$$

$$\xi_2 = \pi_{20} = \rho p_{13} \quad \xi_2 = \pi_{21} = \rho p_{30}$$

$$\xi_3 = \pi_{30} = \rho p_{21} \quad \xi_3 = \pi_{31} = \rho p_{02}$$

in A_2 ; and in A_3

$$\xi_0 = \pi_{02} = \rho p_{31} \quad \xi_0 = \pi_{03} = \rho p_{12}$$

$$\xi_1 = \pi_{12} = \rho p_{03} \quad \xi_1 = \pi_{13} = \rho p_{20}$$

$$\xi_3 = \pi_{32} = \rho p_{10} \quad \xi_2 = \pi_{23} = \rho p_{01}$$

Substituting these values in (2) and also substituting ξ_{ij} for ξ_k , we

get:

$$\begin{aligned}
 (26) \quad & \rho^2 [l_{12} l_{02} p_{12} p_{20} \cos(l_{12} l_{02}) \\
 & + l_{02} l_{01} p_{20} p_{01} \cos(l_{02} l_{01}) + l_{12} l_{10} p_{01} p_{12} \cos(l_{12} l_{10}) \\
 & + l_{02} l_{30} p_{30} p_{02} \cos(l_{02} l_{30}) + l_{30} l_{23} p_{23} p_{30} \cos(l_{30} l_{23}) \\
 & + l_{02} l_{23} p_{02} p_{23} \cos(l_{02} l_{23}) + l_{31} l_{10} p_{31} p_{10} \cos(l_{31} l_{10}) \\
 & + l_{03} l_{31} p_{03} p_{31} \cos(l_{03} l_{31}) + l_{10} l_{03} p_{10} p_{03} \cos(l_{10} l_{03}) \\
 & + l_{32} l_{21} p_{32} p_{21} \cos(l_{32} l_{21}) + l_{13} l_{32} p_{13} p_{32} \cos(l_{13} l_{32}) \\
 & + l_{21} l_{13} p_{21} p_{13} \cos(l_{21} l_{13}) + l_{12} l_{03} p_{12} p_{03} \cos(l_{12} l_{03}) \\
 & + l_{20} l_{13} p_{20} p_{13} \cos(l_{20} l_{13}) + l_{01} l_{23} p_{01} p_{23} \cos(l_{01} l_{23}) \\
 & - l_{12}^2 p_{12}^2 - l_{20}^2 p_{20}^2 - l_{01}^2 p_{01}^2 - l_{03}^2 p_{03}^2 - l_{13}^2 p_{13}^2 - l_{23}^2 p_{23}^2] = 0.
 \end{aligned}$$

now let us express equation (26) in p coordinates. The line equation is, in determinant form,

$$\begin{vmatrix}
 A_{00} & A_{01} & A_{02} & A_{03} & x_0 & y_0 \\
 A_{01} & A_{11} & A_{12} & A_{13} & x_1 & y_1 \\
 A_{02} & A_{12} & A_{22} & A_{23} & x_2 & y_2 \\
 A_{03} & A_{13} & A_{23} & A_{33} & x_3 & y_3 \\
 x_0 & x_1 & x_2 & x_3 & 0 & 0 \\
 y_0 & y_1 & y_2 & y_3 & 0 & 0
 \end{vmatrix} = 0,$$

This is to be developed in terms

of the minors of $\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{vmatrix}$

and gives an expression consisting of:

(29) $\left\{ \begin{aligned} &12 \text{ terms like } + (A_{03}A_{13} - A_{01}A_{33})p_{12}p_{20} \\ &+ (A_{13}A_{23} - A_{12}A_{33})p_{20}p_{01} \end{aligned} \right.$

plus 3 terms like $2(A_{01}A_{23} - A_{02}A_{13})p_{12}p_{03}$
 $+ 2(A_{03}A_{12} - A_{01}A_{23})p_{20}p_{13}$

plus 6 terms like $(A_{03}^2 - A_{00}A_{33})p_{12}^2$
 $= 0.$

Now $l_{02}^2 l_{12}^2 \sin^2(l_{02} l_{12}) = l_{02}^2 l_{12}^2 [1 - \cos^2(l_{02} l_{12})] = 4 \Lambda^2 = 4 A_{33}$

$\therefore l_{02} l_{12} \cos(l_{02} l_{12}) = \sqrt{l_{02}^2 l_{12}^2 - 4 A_{33}}$

and similarly, expressions can be derived for the quantities, $l_{ij} l_{ik} \cos(l_{ij} l_{ik})$.

These expressions substituted in (28) give

(30) $\left\{ \begin{aligned} &12 \text{ terms like } \sqrt{l_{12}^2 l_{20}^2 - 4 A_{33}} p_{12} p_{20} \\ &+ \sqrt{l_{20}^2 l_{01}^2 - 4 A_{33}} p_{20} p_{01} \end{aligned} \right.$

plus 3 terms like $l_{12}l_{03} \cos(l_{12}l_{03}) p_{12}p_{03}$
 plus 6 terms like $-l_{12}^2 p_{12}^2 \quad \quad \quad] = 0$

Now these equations, (29) and (30), represent the absolute and are both written in terms of the same coordinates and consequently their coefficients, term for term, must be the same. The equating of these 21 coefficients will give all the relations existing between the faces and edges of T .

We shall next determine the value of P .

Take the two faces, A_i and A'_j meeting in the edge l_{ij} . Let

h be the altitude of T , to the face A'_j ; and H be the altitude, from the same vertex, of the face A'_i . Let V be the volume of T , and R the sum of

the foot of H and the foot of H .

$$\text{Then } 3V = HA'_j \text{ and } 2A'_i = \ell_{ij} H$$

The triangle HRA is a right triangle and moreover the dihedral angle at ℓ_{ij} is measured by the plane angle between R and H . Consequently

$$p = H \sin(\ell_{ij})$$

These last three equations give

$$\frac{3V}{2} \ell_{ij} = A_i A_j \sin(\ell_{ij})$$

Equating the last coefficients of (29) and (30) we get :

$$p^2 \ell_{ij}^2 = A_{ii} A_{jj} - A_{ij}^2$$

As $A_{ij} = A_i A_j \cos(\ell_{ij})$ we have :

$$p^2 \ell_{ij}^2 = A_{ii} A_{jj} [1 - \cos^2(\ell_{ij})] = A_{ii} A_{jj} \sin^2(\ell_{ij})$$

$$\therefore p \ell_{ij} = A_i A_j \sin(\ell_{ij})$$

$$\text{And } p \ell_{ij} = \frac{3V}{2} \ell_{ij} \text{ or } p = \frac{3V}{2}$$

Equating now the 21 coefficients above and substituting this value of p we have the following 21

relations between the faces and edges of Π .

$$\left(\frac{3V}{2}\right)^2 l_{12}^2 = A_{00} A_{33} - A_{03}^2$$

$$\left(\frac{3V}{2}\right)^2 l_{20}^2 = A_{11} A_{33} - A_{13}^2$$

and four similar relations.

$$\left(\frac{3V}{2}\right)^4 l_{12}^2 l_{20}^2 = (A_{03} A_{13} - A_{01} A_{33})^2 + 4 A_{33} \left(\frac{3V}{2}\right)^4$$

$$\left(\frac{3V}{2}\right)^4 l_{20}^2 l_{01}^2 = (A_{13} A_{23} - A_{12} A_{33})^2 + 4 A_{33} \left(\frac{3V}{2}\right)^4$$

$$\left(\frac{3V}{2}\right)^4 l_{01}^2 l_{12}^2 = (A_{03} A_{23} - A_{02} A_{33})^2 + 4 A_{33} \left(\frac{3V}{2}\right)^4$$

and nine similar relations

$$\left(\frac{3V}{2}\right)^2 l_{12} l_{03} \cos(l_{12} l_{03}) = 2(A_{01} A_{23} - A_{02} A_{13})$$

$$\left(\frac{3V}{2}\right)^2 l_{02} l_{13} \cos(l_{02} l_{13}) = 2(A_{03} A_{12} - A_{01} A_{23})$$

$$\left(\frac{3V}{2}\right)^2 l_{01} l_{23} \cos(l_{01} l_{23}) = 2(A_{02} A_{13} - A_{03} A_{12})$$

The last 3 equations may be written without the cosine function, by use of the following relations given by Ferrers.*

$$2 l_{12} l_{30} \cos(l_{12} l_{30}) = (l_{01}^2 + l_{23}^2) - (l_{02}^2 + l_{13}^2)$$

(and a similar ones.)

making this substitution, the above 3 become

$$\left(\frac{3V}{2}\right)^2 [(l_{01}^2 + l_{23}^2) - (l_{02}^2 + l_{13}^2)] = 4(A_{01} A_{23} - A_{13} A_{20})$$

* Quarterly Journal Vol. 3, p 145.

$$\left(\frac{3\nu}{2}\right)^2 \left[(\ell_{12}^2 + \ell_{03}^2) - (\ell_{01}^2 + \ell_{23}^2) \right] = 4(A_{03}A_{12} - A_{01}A_{23})$$

$$\left(\frac{3\nu}{2}\right)^2 \left[(\ell_{02}^2 + \ell_{13}^2) - (\ell_{12}^2 + \ell_{03}^2) \right] = 4(A_{02}A_{13} - A_{03}A_{12})$$

These relations enable us to express all the constants in the equations of this paper in terms of either the mass of the nucleus, or the energy or the sides of the tetrahedron of reference.

$$\Delta = \begin{vmatrix} x^2 + y^2 + z^2 & x & y & z \\ x & x^2 + y^2 + z^2 & y & z \\ x & y & x^2 + y^2 + z^2 & z \\ x & y & z & x^2 + y^2 + z^2 \end{vmatrix}$$

$$= (x^2 + y^2 + z^2)^3 - 3(x^2 + y^2 + z^2)(x^2y^2 + y^2z^2 + z^2x^2) + 6xyz^2$$

$$= (x^2 + y^2 + z^2)(x^2 + y^2 + z^2 - 3(x^2y^2 + y^2z^2 + z^2x^2) + 6xyz^2)$$

$$= (x^2 + y^2 + z^2)(x^2 + y^2 + z^2 - 3(x^2y^2 + y^2z^2 + z^2x^2) + 6xyz^2)$$

and $x - t = b d + a e - b r - a s$

$$t - c = e r + d s - b l - a e$$

$$c - f = b r + a s - e r - d s$$

$$\Delta \equiv (a^2 + b^2)(e^2 r^2 + d^2 s^2) + (d^2 + e^2)(a^2 s^2 + b^2 r^2) + (r^2 + s^2)(b^2 l^2 + a^2 e^2) = 0$$

$\therefore$ These 4 points are on a circle in the plane $z=0$ and consequently the 4 vertices of Π are on a circular cylinder perpendicular to the plane of the triangle. The line of intersection of the plane $z=0$ and the cylinder is a circle. The 4 points are on this circle.

The section of the cylinder is a circle.

in the z -axis. Thus from any given tetrahedron as a tetrahedron of reference, which is self-holar with regard to an orthic paraboloid, we get two others, T' and T'' , each inscribed to T

and a fourth, T''' , which is the reflection of T in the z -axis, T' in the y -axis, and T'' in the x -axis. Each of the four tetrahedra are inscribed in a cylinder whose generators are perpendicular to the plane that is tangent to the quadric at its vertex. Moreover the axes of these 4 cylinders are on a fifth cylinder whose axis is the z -axis, and these 5 cylinders meet the plane W in the cubic curve,

$$(A^2_y) = 0_x$$

That a cylinder should contain t points, ^{is} in general, only $3 \times t - 4$ or

8 conditions. is the number of conditions. Beltrami has proven that the axes of the cylinders, or 4 points, intersect the line t in t points. This cubic curve is the edge of the cone. Beltrami has

It is not necessary to solve
 the equation in θ for $\theta = 0$
 and $\theta = 180^\circ$ to find the
 values of θ for which $\theta = 0$
 and $\theta = 180^\circ$ are solutions.

J. B. Eck has shown this surface to be
 also of the same degree as Ω , i.e.
 a ruled surface of the ninth degree, and
 has discussed this surface in some
 detail.*

* "Über die Verteilung der Iven der Potions-
 flächen 2. Grades, welche durch 7 gegebene
 Punkte gehen." Königlich Preussische
 zu Münster, 1890. (L. 10. 1. 1.)

three
 1 2 3 4 5 6 7 8 9 10 11 12

This ruled surface is definitely connected with Ω . They are both of the same degree and there is a direct correlation between the generators of the first and those planes of the second that contain perpendicular generators. For each such plane of Ω there is one generator of the first surface that is perpendicular to it.

The axes of the paraboloids of this paper also form a ruled surface, whose degree can not be more than nine, as the generators of it all pass thru the sextic, (14), and the cubic curve, $(\frac{A^2}{y}) = 0$, at infinity.

The axes of the cylinders on the tetrahedra π' , π'' , and π''' , are on ruled surfaces and these last four surfaces all pass thru the cubic curve $(\frac{A^2}{y}) = 0$.

§9. The quadratic transformation.

The equation (1) represents a transformation, that is, a quadratic. Let us write these equations as:

$$31 \quad X_i = C_i Y_i = x_{ij} Y_{ij} + x_{ik} Y_{ik} + x_{il} Y_{il}$$

where $i \neq j \neq k \neq l$ and $i, j, k, l = 0, 1, 2, 3$.

Let us consider the correspondence that establishes between two spaces, S_x and S_y . Substituting these expressions in $\{X_i = 0\}$, we have the correspondence in the S_y :

$$32 \quad \{C_i Y_i\} = 0, \text{ or } \sum_0^3 \{x_{ij} Y_{ij} + x_{ik} Y_{ik} + x_{il} Y_{il}\} = 0.$$

This gives for a plane π in S_x , a quadric in S_y ; for a line in S_x , a quartic curve in S_y ; and consequently for a point in S_x , 8 pts. in S_y . All planes of S_x give a 3-fold system of quadrics in S_y .

a plane in S_y meets a quartic curve in 4 points and consequently, a line in S_x meets the correspondent of a plane in S_y , in 4 points. Therefore, this correspondent is a quartic surface and can be shown to be a Steiner's quartic surface. To a line in S_y corresponds a conic in S_x . a plane ϕ meets the 3-fold system of quadrics in a 3-fold system of conics which map the plane ϕ onto its corresponding Steiner's quartic surface.

The Jacobian of the system of quadrics is a quartic surface and contains the ten lines that are the intersections of the planes that represent the ten degenerate quadrics of the system.

The Jacobian of the system of

quadrics, is:

$$\begin{vmatrix} x_0 y_1 + x_0 y_2 + x_0 y_3 & x_0 y_0 & x_0 y_0 & x_0 y_0 \\ x_0 y_1 & x_0 y_0 + x_1 y_2 + x_1 y_3 & x_1 y_1 & x_1 y_1 \\ x_0 y_2 & x_1 y_2 & x_0 y_0 + x_1 y_1 + x_2 y_3 & x_2 y_2 \\ x_0 y_3 & x_1 y_3 & x_2 y_3 & x_0 y_0 + x_1 y_1 + x_2 y_2 \end{vmatrix} = 0$$

or

$$\begin{aligned} (32) \quad &= x_0 x_1 x_2 x_3 y_0 y_1 y_2 (x_0 y_0 + x_1 y_1 + x_2 y_2) \\ &+ x_0 x_1 x_3 x_2 y_0 y_1 y_3 (x_0 y_0 + x_1 y_1 + x_2 y_2) \\ &+ x_0 x_2 x_3 x_1 y_0 y_2 y_3 (x_0 y_0 + x_1 y_1 + x_2 y_2) \\ &+ x_1 x_2 x_3 x_0 y_1 y_2 y_3 (x_0 y_0 + x_1 y_1 + x_2 y_2) = 0 \end{aligned}$$

$$\text{or, } g \equiv \sum_{i,j,k} x_{ij} x_{ik} x_{jk} y_i y_j y_k (x_{il} y_l + x_{jl} y_j + x_{kl} y_k) = 0$$

Since to a line in S_x , corresponds an elliptic quartic in S_y , the correspondent of g will meet the line in the same number of points, as a quartic curve meets a surface of order four, or sixteen points. Then the correspondent of g is a surface of order sixteen.

The equation of the correspondent of $\mathcal{F}$ will be given in planes, as the discriminant of (32), for a plane ξ will touch this correspondent, when $\mathcal{F}$ touches the quadrics, i.e. when the discriminant vanishes.

Thus, ξ is

$$\begin{vmatrix} 0 & d_{01} & d_{02} & d_{03} \\ d_{01} & 0 & d_{12} & d_{13} \\ d_{02} & d_{12} & 0 & d_{23} \\ d_{03} & d_{13} & d_{23} & 0 \end{vmatrix} = 0 \quad \text{where } d_{01} \text{ represents } d_{01}(\xi_0 + \xi_1), \text{ etc.}$$

$$\text{or } d_{01}^2 d_{23}^2 + d_{02}^2 d_{13}^2 + d_{03}^2 d_{12}^2 - 2(d_{01}d_{02}d_{13}d_{23} + d_{01}d_{03}d_{12}d_{23} + d_{02}d_{03}d_{12}d_{13}) = 0$$

$$\text{or } (d_{01}d_{23} + d_{02}d_{13} - d_{03}d_{12})^2 = 4d_{01}d_{02}d_{13}d_{23}$$

This is at once seen to be

$$(34) \quad Q = \sqrt{d_{01}d_{23}(\xi_0 + \xi_1)(\xi_2 + \xi_3)} + \sqrt{d_{02}d_{13}(\xi_0 + \xi_2)(\xi_1 + \xi_3)} + \sqrt{d_{03}d_{12}(\xi_0 + \xi_3)(\xi_1 + \xi_2)} = 0.$$

This is seen to be of the 4th degree in ξ and therefore the correspondent

of 4 is a surface of class 7

As the reflection of the plane, $\xi_0 + \xi_1 = 0$, in the centroid of T, whose coordinates are $(1,1,1)$ is $\xi_2 + \xi_3$, and similarly for $\xi_0 + \xi_2$ and $\xi_0 + \xi_3$ which reflect into $\xi_1 + \xi_3$ and $\xi_1 + \xi_2$, we see that

The surface 2 is symmetric with regard to the centroid of the tetrahedron of reference.

Let us now consider the correspondence between a plane γ and its corresponding Steiner's cubic surface. In particular, let this plane be the plane, W , at infinity. The equation of this surface is found at once by forming the session of (34), and after this equation is changed over into the point equation, make the surface touch the plane V .

This gives in determinant form:

$$\begin{vmatrix} 0 & x_{01} & x_{02} & x_{03} & 1 \\ x_{01} & 0 & x_{12} & x_{13} & 1 \\ x_{02} & x_{12} & 0 & x_{23} & 1 \\ x_{03} & x_{13} & x_{23} & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

where as before x_{0i} represents $x_{0i}(\xi_0 + \xi_1)$ etc.
This developed out as

$$(35) \quad R \equiv (x_{12} + x_{13} - x_{23})(-x_{01}x_{23} + x_{02}x_{03}) \\ + (x_{12} - x_{13} + x_{23})(x_{02}x_{13} + x_{01}x_{03})$$

$$\begin{aligned}
 &+(-\lambda_{12}+\lambda_{13}+\lambda_{23})(\lambda_{03}\lambda_{12}+\lambda_{01}\lambda_{02}) \\
 &=\lambda_{01}^2\lambda_{23}+\lambda_{02}^2\lambda_{13}+\lambda_{03}^2\lambda_{12}+\lambda_{12}\lambda_{13}\lambda_{33}.
 \end{aligned}$$

The cubic C_3 , i.e. the intersection of W and Ω , maps into a sextic curve on R , the sextic (19), which is a part of the double-point curve on Ω .

The quadric corresponding to the plane W , whose coordinates are 1, 1, 1, 1, is seen from $(\xi\gamma) = 0$, to be $(C\gamma) = 0$, which is the equation of the sphere circumscribed to T . So this quadric is one of the system, and the plane, W , meets it in the absolute, which, consequently, is one of the conics of the 3-fold system in W . Thus to the plane W , in the space S_3 , corresponds the quadric $(C\gamma) = 0$, in S_4 , and to the plane W , in S_4 , corresponds the sextic surface R in S_3 . To the anti

section of C_3 ($C_3 = 0$), and W , i.e. to W & absolute, will correspond a rational quartic, the intersection of W and R .

The absolute meets the cubic C_3 in 6 points and therefore this rational quartic in W will be on 6 points of the sextic W . These two sets of 6 points are one and the same.

To the 4 lines $y_i = 0$, which are the intersections of π and W , will correspond a second set of 4 lines, whose equations are $C_i = 0$, i.e. we

$$l_{01}^2 y_1 + l_{02}^2 y_2 + l_{03}^2 y_3 = 0,$$

$$l_{01}^2 y_0 + l_{12}^2 y_2 + l_{13}^2 y_3 = 0,$$

$$l_{02}^2 y_0 + l_{12}^2 y_1 + l_{23}^2 y_3 = 0,$$

$$l_{03}^2 y_0 + l_{13}^2 y_1 + l_{23}^2 y_2 = 0.$$

Hesse has shown that if T and T' are 2 tetrahedra, self-polar as to a quadric, their vertices are a set of 8 associated points.

The converse is equally true, i.e. ^{which can be that of no 2 tetrahedra} for 8 associated points, there is a ^{tetrahedron} definite quadric to which the _^ are self-polar.

suppose $(LX)^2$, $(MX)^2$, $(NX)^2$ to be 3 quadrics on the 8 points of an associated set, and that ~~for~~ these points form the vertices of the tetrahedron of reference and let us name the other four a, b, c, d . Let (KX^2) be the quadric to which both tetrahedra are self-polar. The polar of x' as to $(KX^2) = 0$, is $(KXX') = 0$ and if a is $(Kax) = 0$. If this plane is to be on b , then $(Kab) = 0$. Similarly $(Kac) = 0$, and $(Kbe) = 0$. These

3 conditions on a, b , and c will make the tetrahedron a, b, c, d self-polar as to $(\lambda x^2) = 0$, for then λ is determined.

Eliminating λ between the last 4 equations we have the quadric $(\lambda x^2) = 0$ in this form:

$$(36) \begin{vmatrix} x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ a_0 b_0 & a_1 b_1 & a_2 b_2 & a_3 b_3 \\ b_0 c_0 & b_1 c_1 & b_2 c_2 & b_3 c_3 \\ c_0 a_0 & c_1 a_1 & c_2 a_2 & c_3 a_3 \end{vmatrix} = 0$$

If $(2X)^2, (3X)^2, (4X)^2$ respectively we on a, b, c , then

$$(37) \begin{cases} d_{01} a_0 a_1 + d_{02} a_0 a_2 + d_{03} a_0 a_3 + d_{12} a_1 a_2 + d_{13} a_1 a_3 + d_{23} a_2 a_3 = 0 \\ 3_{01} b_0 b_1 + 3_{02} b_0 b_2 + & + & + & + = 0 \\ v_{01} c_0 c_1 + v_{02} c_0 c_2 + & + & + & + = 0 \end{cases}$$

From these 3 equations we have

$$(38) \begin{vmatrix} d_{ij} & d_{ik} & d_{il} \\ 3_{ij} & 3_{ik} & 3_{il} \\ v_{ij} & v_{ik} & v_{il} \end{vmatrix} = \rho \begin{vmatrix} a_k a_l & a_l a_j & a_j a_k \\ b_k b_l & b_l b_j & b_j b_k \\ c_k c_l & c_l c_j & c_j c_k \end{vmatrix}$$

or the $d, 3$, and v 's expressed in terms.

of the coordinates of the points a, b, c .
Now

$$(39) \begin{vmatrix} a_k a_l, & a_l a_j, & a_j a_k \\ b_k b_l, & b_l b_j, & b_j b_k \\ c_k b_l, & c_l c_j, & c_j c_k \end{vmatrix} = \begin{vmatrix} a_l b_l, & a_j b_j, & a_k b_k \\ b_k c_l, & b_j c_j, & b_k c_k \\ c_k a_l, & c_j a_j, & c_k a_k \end{vmatrix}$$

By means of these last two relations,
the coefficients of x_i^2 in (36) can be
expressed in terms of α, β and V .
Thus the coefficient of x_0^2 is:

$$\begin{vmatrix} a_1 b_1, & a_2 b_2, & a_3 b_3 \\ b_1 c_1, & b_2 c_2, & b_3 c_3 \\ c_1 a_1, & c_2 a_2, & c_3 a_3 \end{vmatrix} = \begin{vmatrix} a_1 a_3, & a_1 a_2, & a_2 a_3 \\ b_1 b_3, & b_1 b_2, & b_2 b_3 \\ c_1 c_3, & c_1 c_2, & c_2 c_3 \end{vmatrix} = \begin{vmatrix} \alpha_{02}, \alpha_{03}, \alpha_{01} \\ \beta_{02}, \beta_{03}, \beta_{01} \\ \nu_{02}, \nu_{03}, \nu_{01} \end{vmatrix}$$

(36) then becomes:

$$(40) \quad x_0^2 |\alpha_{01} \beta_{02} \nu_{03}| + x_1^2 |\alpha_{01} \beta_{13} \nu_{12}| + x_2^2 |\alpha_{02} \beta_{12} \nu_{23}| \\ + x_3^2 |\alpha_{03} \beta_{23} \nu_{13}| = 0$$

We shall now apply this to the
particular case at hand.

In the special case of this section
the 8 points, y , corresponding to a
given point x , break up into 2 sets

of 4 points each, namely:- the one set that represents the fundamental tetrahedron, and which consequently is fixed and common to all sets of 8 points in S_y that correspond to points in S_x ; the second set of 4 points, which is different for each set of 8.

Let the fixed set be $x_i = 0$. ($i=0,1,2,3$)
 Then if y be one of the remaining set of 4 points, and y' any one of the remaining 3, there would be these relations, arising from $x_i = c_i y_i$

$$\frac{c_0 y_0}{c_0' y_0'} = \frac{c_1 y_1}{c_1' y_1'} = \frac{c_2 y_2}{c_2' y_2'} = \frac{c_3 y_3}{c_3' y_3'}$$

From these 3 equations, 3 quadratics arise:

$$(41) \begin{cases} c_0 c_1' y_0 y_1' = c_0' c_1 y_0' y_1 \\ c_0 c_2' y_0 y_2' = c_0' c_2 y_0' y_2 \\ c_0 c_3' y_0 y_3' = c_0' c_3 y_0' y_3 \end{cases}$$

which, if we call $\frac{c_{ij}^{(1)}}{c_{ii}^{(1)}} = A_i$, become

$$c_{0i}' \gamma_{0i}' = A_1 c_{1i}' \gamma_{1i}' = A_2 c_{2i}' \gamma_{2i}' = A_3 c_{3i}' \gamma_{3i}'.$$

or, if we write γ_{0i}' for $\gamma_{0i}' \gamma_{1i}'$,

$$(42) \left\{ \begin{aligned} l_{01}^2 \gamma_{01}' + l_{02}^2 \gamma_{02}' + l_{03}^2 \gamma_{03}' &= A_1 (l_{01}^2 \gamma_{01}' + l_{12}^2 \gamma_{12}' + l_{13}^2 \gamma_{13}') \\ &= A_2 (l_{02}^2 \gamma_{02}' + l_{12}^2 \gamma_{12}' + l_{23}^2 \gamma_{23}') \\ &= A_3 (l_{03}^2 \gamma_{03}' + l_{13}^2 \gamma_{13}' + l_{23}^2 \gamma_{23}') \end{aligned} \right.$$

These 3 quadrics all pass thru the vertices of T and thru the other set of 4 points, T' . To get the quadric with regard to which, T & T' are self-polar, we will only have to substitute in (40) the following values for α, β , and γ .

$$\begin{aligned} \alpha_{01} &= l_{01}^2 (1 - A_1) & \alpha_{02} &= l_{02}^2 & \alpha_{03} &= l_{03}^2 \\ \beta_{01} &= l_{01}^2 & \beta_{02} &= l_{02}^2 (1 - A_2) & \beta_{03} &= l_{03}^2 \\ \gamma_{01} &= l_{01}^2 & \gamma_{02} &= l_{02}^2 & \gamma_{03} &= l_{03}^2 (1 - A_3) \\ \alpha_{12} &= -l_{12}^2 A_1 & \alpha_{13} &= -l_{13}^2 A_1 & \alpha_{23} &= 0 \\ \beta_{12} &= -l_{12}^2 A_2 & \beta_{13} &= 0 & \beta_{23} &= -l_{23}^2 A_2 \\ \gamma_{12} &= 0 & \gamma_{13} &= -l_{13}^2 A_3 & \gamma_{23} &= -l_{23}^2 A_3 \end{aligned}$$

The result of this substitution is :

$$\begin{aligned} & l_{01}^2 l_{02}^2 l_{03}^2 (A_1 A_2 + A_1 A_3 + A_2 A_3 - A_1 A_2 A_3) X_0^2 \\ & + l_{01}^2 l_{12}^2 l_{13}^2 (A_1 A_2 + A_1 A_3 - A_2 A_3 + A_1 A_2 A_3) X_1^2 \\ & + l_{02}^2 l_{12}^2 l_{23}^2 (A_1 A_2 + A_2 A_3 - A_1 A_3 + A_1 A_2 A_3) X_2^2 \\ & + l_{03}^2 l_{13}^2 l_{23}^2 (A_1 A_3 + A_2 A_3 - A_1 A_2 + A_1 A_2 A_3) X_3^2 = 0 \end{aligned}$$

Putting in the values of the A 's, the result reduces to :

$$(4_3) \left\{ \begin{aligned} & l_{01}^2 l_{02}^2 l_{03}^2 (l_{13}^2 \gamma_1 \gamma_3 + l_{23}^2 \gamma_2 \gamma_3 + l_{12}^2 \gamma_1 \gamma_2) X_0^2 \\ & + l_{01}^2 l_{12}^2 l_{13}^2 (l_{02}^2 \gamma_0 \gamma_2 + l_{03}^2 \gamma_0 \gamma_3 + l_{23}^2 \gamma_2 \gamma_3) X_1^2 \\ & + l_{02}^2 l_{12}^2 l_{23}^2 (l_{01}^2 \gamma_0 \gamma_1 + l_{03}^2 \gamma_0 \gamma_3 + l_{13}^2 \gamma_1 \gamma_3) X_2^2 \\ & + l_{03}^2 l_{13}^2 l_{23}^2 (l_{01}^2 \gamma_0 \gamma_1 + l_{02}^2 \gamma_0 \gamma_2 + l_{12}^2 \gamma_1 \gamma_2) X_3^2 = 0. \end{aligned} \right.$$

This gives, at once, a relation between the points γ and γ' , which we may take of the second set of 4 points. This relation is of such a nature that one γ' determines 3 γ 's, for this point $\gamma' = x_\gamma$ determines a point x_γ in S_x which in turn determines the 4 points of which this γ' is one, or it determines the other 3 points of the set.

is then 4 points
 will lie
 in the polar plane of this fourth point.
 If the plane of the 3 points, y , is taken
 as η , its coordinates will be:

$$\begin{aligned}\eta_0' &= \Delta_0 \Delta_2 \Delta_3 (\Delta_{13} \gamma_1 \gamma_3 + \Delta_{23} \gamma_2 \gamma_3 + \Delta_{12} \gamma_1 \gamma_2) \gamma_0 \\ \eta_1' &= \Delta_0 \Delta_1 \Delta_3 (\Delta_{02} \gamma_0 \gamma_2 + \Delta_{03} \gamma_0 \gamma_3 + \Delta_{23} \gamma_2 \gamma_3) \gamma_1 \\ \eta_2' &= \Delta_0 \Delta_1 \Delta_2 (\Delta_{01} \gamma_0 \gamma_1 + \Delta_{03} \gamma_0 \gamma_3 + \Delta_{13} \gamma_1 \gamma_3) \gamma_2 \\ \eta_3' &= \Delta_0 \Delta_2 \Delta_3 (\Delta_{01} \gamma_0 \gamma_1 + \Delta_{02} \gamma_0 \gamma_2 + \Delta_{12} \gamma_1 \gamma_2) \gamma_3\end{aligned}$$

This is a cubic transformation of
 the γ 's.

For we make y' incident with
 η , i.e. $(\eta y') = 0$, we get the equation
 of the Jacobian as given in (33), to
 within a factor. Also if in (43) we
 make x coincident with y , we
 get at once get the equation of
 the Jacobian of the net of quadrics.

Vita

Harry Clinton Gossard was born at Helena, Idaho on March 13 1884. He was prepared for college in the public schools, and graduated from Idaho Northern University with the degree of Bachelor of Science in July 1907. In October 1910, he entered the Johns Hopkins University, as a candidate for the degree of Doctor of Philosophy, with Mathematics as his principal subject and astronomy and Education as first and second subordinates. He held a University Scholarship during the year 1911-1912 and a University Fellowship during the year 1912-1913.